APPROACHING THE DISTRIBUTIVE LAW WITH YOUNG PUPILS

Nicolina A. Malara and Giancarlo Navarra

This paper contributes to the research strand concerning early algebra and focuses on the distributive law. It reports on a study involving pupils aged 8 to 10, engaging in the solution of purposefully designed problem situations. These situations are organized to favor specifying the students' solutions and to motivate a collective comparison of the arithmetic expressions that codify the solution processes. The study focuses on ways in which perception leads to different mental images that influence the choice of either the $(a+b)\times c$ *or the* $(a\times c)+(b\times c)$ *representation. It highlights that understanding these dynamics is a fundamental step for a meaningful learning of the property.*

Keywords: Distributive law; Language representation of processes; Mental models; Perception

Aproximación a la Propiedad Distributiva con Estudiantes Jóvenes

Este artículo contribuye a la rama de investigación relativa al early algebra *y se centra en la propiedad distributiva. Describimos un estudio que involucra estudiantes de 8 a 10 años, implicados en la resolución de problemas. Estos problemas se han organizado para favorecer un enunciado explícito de las soluciones propuestas por los alumnos y motivar una comparación colectiva de expresiones aritméticas que codifican los procesos de resolución. El estudio se centra en las formas en las que la percepción da lugar a diferentes imágenes mentales que llevan a elegir la representación* $(a+b)\times c$ *o* $(a\times c)+(b\times c)$. *La comprensión de esta dinámica es un paso fundamental para un aprendizaje significativo de la propiedad.*

Términos clave: Lenguaje de representación de procesos; Modelos mentales; Percepción; Propiedad distributiva

This work is part of the ArAl project, which was designed to revisit arithmetic teaching in a pre-algebraic perspective (Malara & Navarra, 2003a, 2003b, 2004)

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and concerns a fragment of a teaching path centered on problem solving activities. The purpose of these activities is to construct in pupils an experiential basis for an objectification of the distributive law, through collective discussions that promote sharing and reflection. The theoretical frame of the work is essentially the one of the project[1] and it is sketched in the quoted papers. The distributive law, together with the associative and commutative laws, plays a key role on both the arithmetical —mental calculations, algorithms, rule of signs, etc.— and the algebraic sides —transformation of expressions, recognition of equivalence relationships, formal identities, etc.— and, more generally, in the production of thinking via algebraic language. In usual teaching practice, however, these properties are taken for granted, almost assumed as tacit axioms, or worse, they are assigned to be learned by heart from the textbook. Pupils are thus led to the position not to understand the sense of these properties, to perceive a rupture between the experiential and the theoretical, and not to recognize their value on the operative level. The tacit spreading of this phenomenon is documented by studies concerning teachers (Tirosh, Hadass, & Movshovich-Hadar, 1991) and by studies focusing on a conscious learning of arithmetical properties and of the distributive law in particular (Mok, 1996; Vermeulen, Olivier, & Human, 1996). In our project this property enters the game in many situations. It is exactly due to this pervasiveness that we deemed important to design a path aimed at its objectification through problem situations that highlight its genesis. Our first results highlight the influence of perception on the construction of mental images useful for conceptualizing the property, and the effectiveness of processes of sharing.

The Situation

The activities were performed in a grade 4 class from Birbano (Belluno, Italy) at the beginning of the school year 2002-03. They were planned within a yearly cycle of meetings with teacher/researchers Giancarlo Navarra and Cosetta Vedana. They were teachers in the mathematical-scientific area and they were simultaneously present. The class was beginning a path which would lead to the conceptual embryo of the distributive law.

The objective was to construct premises for subsequent developments, with the purpose of consolidating the property as a mathematical object. The activities were developed through three problem situations favoring a dynamic development that can be summarized in three phases:

- from confusion to the first arithmetical representations;
- from the first perceptions to the two constitutive representations of the property; and
- reflection on the two representations and appropriation of the mutual equality of the expression values.

[1] http://www.aralweb.unimore.it

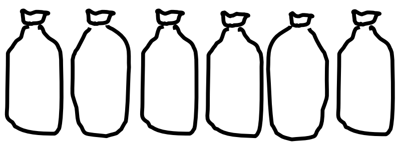
The three situations were meant to favor the transparency of the transition from *perception* of the situation to *translation* into mathematical language. It is thus necessary to: (a) Educate pupils' perception —i.e., lead them to become aware of the existence of diverse ways to perceive a situation, among which some may be more productive from a mathematical point of view—, and (b) make pupils understand that it is possible —through collective sharing— to understand the meaning of translations and conceptualize their mutual equivalence beyond the process each of them identifies. Very often teachers themselves must be educated analogously.

We present here a teaching sequence, overall lasting about three hours, distributed in three sessions, to be considered as an example of the evolution of thinking in both individual and collective forms. The most meaningful parts of the diary are described in detail, whereas other parts, meaningful for their overall sense, are synthesized. As the reader will notice, the initial interventions by pupils denote certain confusion about the assigned task and an apparent regression with respect to the competencies that were acquired the previous year in the solution of problems that had similar structures. These are consequences of the fact that the assignment repeatedly asked them not to solve the problem of finding a result, but to explore one's own modus operandi. Confusion is thus due to an atypical didactical contract: Pupils were asked to work at the metacognitive level, and this request, although having strong educational value, is harder to manage by both students and teacher. One of the main features of the ArAl project is to favor reflection on processes. For this purpose, we promoted activities that stimulate metacognitive and metalinguistic competencies and construct sensitivity towards these aspects in teachers.

From Confusion to the First Arithmetical Representations

1. The activities started with the following assignment:

> *The teacher puts 6 bags —made of a non-transparent fabric— on a desk and explains that each of them contains 7 triangles and 12 squares.*



The task is to count how many objects the bags contain. It is strongly underlined that the important thing is not the number of objects, but rather the reasoning process followed to find it. Pupils know the quantities referred to objects but cannot see them. Therefore they are forced to construct mental models. In order to do this, they must initially focus on their perception of the imagined situation, in an intertwining of unstable perceptions and floating calculation attempts.

The task is complex. In fact, the pupil is asked not to count, but to look at himself or herself in the counting act. He/she must face a metacognitive task: Reflecting on his own actions.

From Confusion…

2. The first difficulty is a psychological one. Pupils were anxious and did not understand the task.
3. The next step occurred at the cognitive level. Being uncertain about the task, pupils went for the most familiar interpretation and mentally counted the content of a bag: "There are 19 items". Pupils were still not seeing the items, but they needed to imagine them. Therefore, counting was done on a virtual context, without the reassuring feedback given by a physical contact with objects. But, as we will see later, pupils kept interpreting the request to count in a strict sense, an operative one, instead of managing a complex situation in which counting could become a sort of umbrella, under which several strategies for calculation could be developed. What seemed to be important for them was the number, i.e., the result.
4. The teacher gave a visual aid (see Figure 1) and he invited the pupils to open the bags.

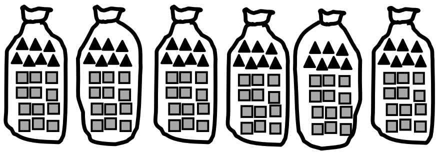


Figure 1. Visual aid given by the teacher

5. There was still confusion. Seeing the objects did not seem to offer significant help to pupils. While searching for an interpretation of the task, they started manipulating the items. Some of them grouped the items by color and shape, while others left them shuffled. However, this manipulation did not provide them with particular hints.
6. The teacher reformulated the task and asked the pupils for a discussion: "I did not ask you to tell me a number… do you remember? I asked you to count mentally the pieces and then try to explain how you proceeded for counting. Look inside yourselves, as in a movie. What did you think? Where did you start from?"
7. In the new discussion there was still confusion. Nevertheless, the discussion was more choral and animated, with weak metacognitive features. Some pupils said that they counted the items one by one —as we underlined previously, counting still emerges as a litany—, others used the times 2-table, others used the times 5-table —the discussion about the strategies highlighted

the fact that they counted pieces two or five at a time, to go faster—, and others counted by groups of colors.

8. The teacher formulated the task again, and he invited pupils to give less generic explanations, taking into account the information given by the problem: "Look carefully at what is in front of you. There are six bags. Each bag has the same content, made of triangles and squares, and there are 7 triangles and 12 squares. Your brains are working with these numbers".
9. The activity evolved at a metacognitive level. Pupils, working in small groups, manipulated blocks meditating on the moves, with slow shifts accompanied by reflection. In a Gestaltian sense, pupils were *restructuring their field*, searching for meaningful perceptions. Calculation processes started shaping up in a complete and communicable way. Embryos of processes were proposed. For instance, pupils of a group answered that, in order to find the total number, they did 19 times 6.

Steps 1-9

The initial situation (1) in which triangles and squares were not visible made pupils uncomfortable (2) and was sorted out by means of a calculation (3). Nevertheless, it forced pupils to construct mental images of the situation. Seeing physically the objects (4) did not help them in the beginning (5) because possibly the real problem did not lie in vision per se, but in the organization of the vision itself. The repeated invitation to look inside oneself (6-8) led to an increasing development of metacognitive activity and, consequently, to the elaboration of more organized attempts to *see* the situation with the eyes of mind. Hence a virtuous circle was enacted (9) between an increasingly guided perception and a growing clarity in the interior visualization of mental processes and in their verbal description.

At this stage, the situation was mature for Brioshi's entry. Brioshi is an imaginary Japanese pupil —variably aged according to the age of his interlocutors— and is a powerful support within the ArAl project —the first unit is completely dedicated to him—. He was introduced to make pupils aged between 7 and 14 approach formal coding and the difficult related concept of needing to respect rules in the use of language. This need is even stronger when engaging with a formalized language, because of the extreme synthetic nature of the symbols used in them. Brioshi is able to communicate only through a correct use of mathematical language and enjoys exchanging problems and solutions with foreign classes, through a wide range of instruments. These instruments include messages written on paper sheets or more sophisticated exchanges through the Internet.

…to the First Arithmetical Representations

10. Mathematical language entered the scene. It was time to verify if and how field restructuring —and hence the game of back-and-forths between perception and development of mental models— had produced images that could be represented through mathematical language. The teacher researchers asked the pupils: “What message could you send Brioshi to explain how you managed to count triangles and squares inside a bag and then to find the total number of triangles and squares?”

11. Part of the pupils formulated —individually— the following proposals, transcribed on the blackboard and then discussed.

    a) $9+7+3=19$

    b) 19×6

    c) $(5+5+5+5+5)$

    d) $5\times 3+4=19$

    e) $5\times 4-1$

    f) $2\times 4=8+11=19$

These sentences highlight a short circuit working on the task. Except for (b), the other sentences reflect a conviction that different ways of expressing the content of a bag, that is 19, must be listed. This misunderstanding leads to substantially unreasonable expressions, often impenetrable, because the pupils cannot justify the reasons underlying their representation.[2]

12. The teacher’s invitation to pupils to use a representation in mathematical language was premature and natural language became again the mediator —with a fundamental role, given the age of pupils— through which pupils were asked to describe the concrete situation as it is. At the end of the discussion, the class came to a collective formulation: “There are six bags, all on a desk: There are seven triangles in each bag and 12 squares in each bag, we must represent and find how many they are altogether.”

13. In order to take into account what they had said, the teacher proposed to send a new message to Brioshi.

14. The class formulated different proposals:

    g) $7+12=19$

    h) $7\times 6+12\times 6$

[2] It often happens that when the task is not clearly understood, pupils that express a higher self-confidence are the least aware whereas more prudent pupils show to have a stronger critical capacity and prefer to wait and see what happens rather than taking part in the discussion.

i) $72+42$

j) $19+19+19+19+19+19$

These expressions showed an evolution with respect to the previous ones. Through discussion, pupils focused on (h), (i) and (j) but they saw them as different things. They did not grasp the underlying mental models.

15. The teacher asked the pupils to re-describe the situation in written natural language.
16. Some descriptions were still generic. For instance, "there are six bags and two different shapes". However, two families of descriptions emerged that marked the beginning of a turning point: (a) "six groups of squares and six groups of triangles" and (b) "six bags, in each bag there are seven triangles and 12 squares".
17. The teacher asked the pupils to write other sentences for Brioshi individually. Two groups of sentences came out, referring to the two models.

$$a\times c+b\times c\quad\begin{cases}\text{a)}\ \ 7\times 6,\ 6\times 12,\ 72+42\\ \text{b)}\ \ 72+42\\ \text{c)}\ \ 6\times 12+6\times 7\end{cases}$$

$$(a+b)\times c\quad\begin{cases}\text{d)}\ \ 19\times 6\\ \text{e)}\ \ 12+7\times 6\\ \text{f)}\ \ (12+7)\times 6\\ \text{g)}\ \ 7+12\times 6\end{cases}$$

18. The teacher asked the pupils to comment on their formulations.
19. At the end of discussion the following conclusions were reached. In the first group, the pupils found the whole lot of triangles and then the whole lot of squares. In the second group, the pupils calculated the number of squares and triangles altogether.

Steps 10-19

Brioshi's entry (10) started up an activity of representation in mathematical language. After a start influenced by a possible misunderstanding on the task (11) the recourse to formalized and natural language alternatively (12-14) produced increasingly meaningful results. A system of relationships is outlined that can be visualized through the model shown in Figure 2.

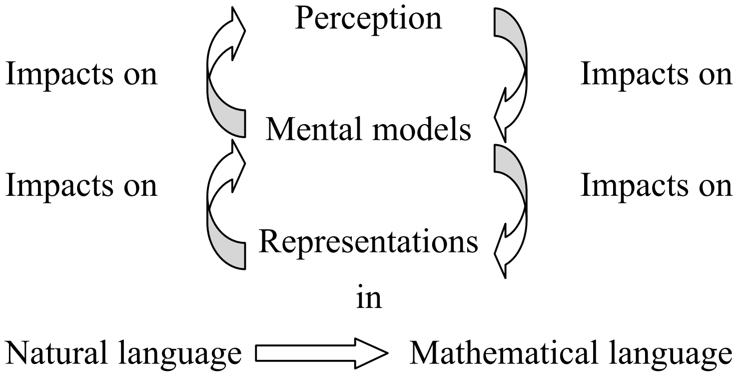
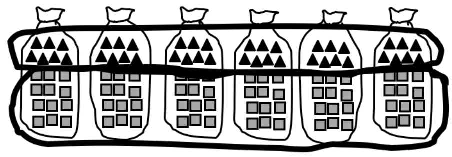


Figure 2. Relationships detected in Brioshi's response

Pupils' increasing capability in moving inside the relationships illustrated in the model led to the production of sentences (h), (i) and (j) in (14). Its transparency made possible to trace back the organization of perceptions that generated them. Pupils were the protagonists of these reconstructions, through which the activity is read at a metacognitive level.

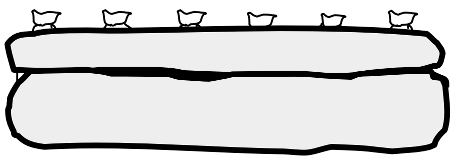
h) $7 \times 6 + 12 \times 6$

The relationship between representation and the mental model elaborating perception is transparent.



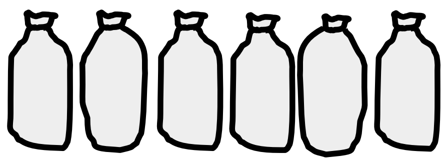
i) $72 + 42$

The representation (R) reflects the previous mental model (M) but it is opaque. The relationship between M and R is weak. Control at metacognitive level is scarce.



j) $19 + 19 + 19 + 19 + 19 + 19$

The representation refers to a perceptive act, and therefore to a mental model, that although different, is still opaque.



Through further intertwining of natural language (15-16), mathematical language (17) and natural language again (19), pupils reached conclusions that introduced effectively an embryo of the distributive law (19).

IMPROVING THE TWO REPRESENTATIONS

20. A week later, the teacher proposed a new problem to the pupils:

Granny prepared for Santa Lucia 8 bags of sweets for her nephews. In each bag she puts 5 chocolates and 14 candies. How many sweets did granny buy? [3]

21. Pupils solved the problem with no questions about clarification. Two types of solutions were provided and they could be ascribed to both representations:

$$a \times c + b \times c \text{ (5 pupils)} \quad \begin{cases} \text{a) } 5 \times 8 = 40 \\ \text{b) } 14 \times 8 = 112 \\ \text{c) } 112 + 40 = 152 \end{cases}$$

$$(a + b) \times c \text{ (6 pupils)} \quad \begin{cases} \text{d) } 14 + 5 = 19 \\ \text{e) } 19 \times 8 = 152 \end{cases}$$

Both models were nearly equally distributed. The proposed calculations were all carried out separately until the result was obtained.

22. During the discussion two pupils provided decisive contributions. Denise wrote in a rough copy $5 \times 8 + 14 \times 8 =$. But she did not know how to continue and preferred to go back to single operations. She explained that she had recognized the same problem she tackled previously. On the other hand, Giada realized that Denise's procedure was the translation of the first solution type and tried to translate the second type solution, writing $14 + 5 + 19 \times 8$. But she realized that it was not good. Collective discussion helped her to modify it till she wrote $(14 + 5) \times 8$.

Steps 20-22

The presentation of a new problem (20) raised two types of representations by separate steps, which can be reduced to those of the distributive law (21). During discussion, representations in a line appeared (22). The former representations were blocking, whereas the latter constituted a fertile ground.

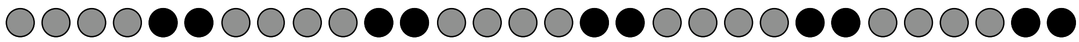
Leading pupils to representations in a line seems to be a necessary condition —although is not a sufficient condition— to construct a mental attitude that may favor the transition to an embryonic view of the property. As we mentioned before, this condition is subordinated to an education to perception of elements of the problem situation. At a first level, most pupils were attracted by aesthetic, formal and expressive aspects that distracted them from the logico-mathematical aspects. Denise and Giada were probably two among the few students that showed a natural inclination for selective analysis. Generally, education plays a determinant role. This means leading the class, through sharing, to make perceptions and reasoning explicit, so that differences may become productive for a collective construction of shared knowledge.

[3] The question "How many sweets…" although focusing on the outcome and not on making the process explicit is nevertheless clear to pupils due to the established contract.

REFLECTING ON THE TWO ARITHMETICAL REPRESENTATIONS

23. A week later, the teacher proposed a third problem situation to the pupils:

A giant cardboard necklace made of alternating four grey beads an two black beads is shown:



Explain in either natural language or mathematical language —or both— the way in which one can find how many beads compose the necklace.

Again pupils must try to describe what they are thinking.

24. Proposals were compared and commented upon:

Giulia I count how many beads are: $2 \times 5 + 4 \times 5$.

But "how" did you count? [note written by the teacher].

I counted this way: the beads are thirty and to make this result I counted[4] them with multiplication.

Lorena I calculate how many black beads and grey beads are: $2 \times 5 + 4 \times 5$.[5]

Claudia For every two black beads there are four grey beads:[6] $2 \times 5 + 4 \times 5$.

Giada Two black ones and then four grey ones:[7] $2 \times 5 + 4 \times 5$.

Alberto $(2 \times 5) + (4 \times 5) = 10 + 20 = 30$. I did two the number of black beads and I multiplied it by 5, same thing for 4.

The class realized that everybody used formulations of a single type, following the first model. The pupils' sentences reveal the dominance in the perception of the black color, which could have lead the pupils to overcome the sequential order of the beads.

25. The teacher invited the pupils to express the situation with the other mathematical formulation. Alberto proposed the following expression, raising general satisfaction in the class: $(2 + 4) \times 5$.

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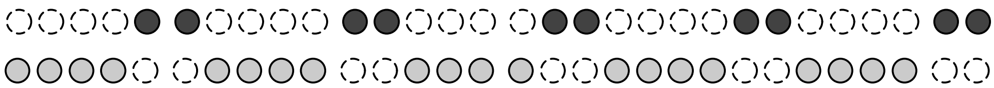
[4] For many young pupils the verb "to count" has a similar meaning to the verb "to calculate". Perhaps to Giulia the two verbs expressed the same action, the same content, and this action and content can be expressed only in mathematical language. Or rather, to her the latter is the most spontaneous way to find the number of beads. The activity is carried out at cognitive and not metacognitive level.

[5] Once she explained what she did, Lorena suggested that numbers and mathematical signs express how you calculate.

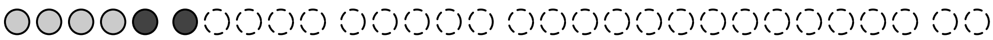
[6] This is the description of what she saw in the necklace. Is it also the description of the way in which she got to the solution?

[7] See previous note.

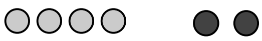
26. The teacher asked the class to explain how the necklace was "viewed" by those who wrote $4\times 5+2\times 5$ and Alberto, who wrote $(2+4)\times 5$.
27. Giulia: "We count how many black beads are and then the grey ones and put them together".[8] Giada: "Alberto adds the four grey beads to the two black beads and repeats them five times."
28. The teacher proposed a mental experiment to the class: "Imagine a completely dark place where you can switch on a spotlight to illuminate the things you want to highlight every time. Your necklace is in this dark place. Draw a sketch showing, as under the spotlight, the necklace seen in the first case, i.e. by the class, and then another sketch showing the necklace in the second way, i.e., seen by Alberto."
29. After some uncertainty, pupils highlighted the two moments in which the necklace was perceived in the first case. They draw two sequences of beads in which they highlight separately the beads of different colors leaving the others white:



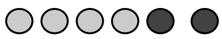
The necklace seen by Alberto needs one single moment in which the spotlight highlights the repeated module.



Stencils and friezes were recalled. Pupils agreed that in the first case, two stencils are needed



and in the second case one stencil was enough.



A pupil said that it was more convenient and recalled the already encountered economy principle.

30. The need for a mathematical expression came back to make Brioshi understand that the two ways of seeing the necklace were equivalent. Pupils proposed quickly the following expression: $4\times 5+2\times 5=(2+4)\times 5$.

Steps 22-30

A third problem (23) lead to representations referring to the only expression $a\times c+b\times c$ (24), although the formulation of the text seemed to induce $(a+b)\times c$. The perception more frequently found confirms what emerges from

[8] Giulia uses the verb to count in an extensive way, condensing more complex actions concerning a sequence of calculations.

other activities of the ArAl Project, concerning the search for regularities. In front of a sequence —frieze, necklace, etc.— characterized by alternating groups of elements, for instance two, pupils identify alternation more regularly than repetition of a module made of both groups. The hypothesis we formulate is that perceiving independent elements is more spontaneous than perceiving relationships between elements. Another hypothesis is that, since seeing is a procedural activity, the diversity of colors breaks the perception of the unity of a module, highlighting two subsequences, and this would induce a distributed vision: $a \times c + b \times c$. The other one, $(a+b) \times c$, is more evolved because it concerns a vision that goes beyond colors and captures the unitary structure of the bicolor module. The two hypotheses are being compared and analyzed in depth. Perception of the alternation hinders the identification of the structure of the sequence and inhibits representation $(a+b) \times c$. A field restructuring, in Gestaltian terms, is necessary.

The teacher's invitation led to the emergence of $(a+b) \times c$ (26) and to a verbal description of the mental models underlying $a \times c + b \times c$ and $(a+b) \times c$ (27-28). An experiment was proposed to favor a re-reading of the context (29). This led the class to elaborate on visualizations that make the two different perceptions transparent (30) and to an intuition of the equality of the two representations.

Conclusions

We now simply give a short indication about the prosecution of the didactic path. The key point is focusing pupils' attention on the comparison of the arithmetical writings arising from the solutions of faced problems, in order to lead them to grasp the general validity of the equality $(a+b) \times c = (a \times c) + (b \times c)$. The main steps of this part of the path are:

- problem situations with iconic support differing for both context and numerical values, in order to favor the two different perceptions of the field;
- problem situations similar to the previous ones, without iconic support that differ for both context and numerical values;
- problem situations proposed in two partially different versions, in order to strengthen the sense of the two representations;
- comparison among problem situations and the related expressions representing their solutions, in order to favor the understanding of the independence of equalities from numerical values and types of data; and
- framing of the various equalities in a scheme and conceptualization of the property.

The detailed analysis of these steps of the path and the reflections about the ways in which the pupils conceptualize the property will be the topic of another paper.

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Nicolina A. Malara
University of Modena and Reggio Emilia
malara@unimo.it

Giancarlo Navarra
University of Modena and Reggio Emilia
giancarlonavarra@gmail.com